Behavioral specifications

Dynamical approach

Draft

Tomasz Nowicki *

Institute of Mathematics, Warsaw University

Banacha 2, 02-097 Warsaw, Poland

`tomnow@mimuw.edu.pl`

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Abstract

We give a description of a *behavioral specification* in terms of infinite sequences. This description allows mathematically precise definitions of needed notions and shows the way to strict proofs of their properties. We propose a handful of such notions and give an overview of their properties. In a part of these notes we show the links with existing theories and point out possible further development of our approach.

1 Introduction

In this note we want to give a mathematical model to the *system behaviour paradigm*. For motivation, general introduction and specific examples we refer to the report [Tur94] which we assume is known to the reader. Here we remind only the general setting.

The system behaviour is defined by actions, which are executed by some agents. An action may be started if some condition (called a preguard) is

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Nowicki, T.

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Institute of Mathematics
Banacha 2
Warsaw 02-097
Poland

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satisfied. Once started the agent acts independently of the system. We assume that each action terminates. Upon a termination of an action another condition (a postguard) is checked and if it is satisfied the result of the action is implemented to the system, otherwise it is abandoned.

Now we outline the model.

- In order to settle the notion of the system we introduce the set of possible states called a world. The behaviour of the system is a sequence of states, so we assume that the changes occur discretely (i.e. not continuously). We shall say earlier or later (i.e. introducing time) in the sense of the precedence in such a sequence. The time is discrete and is modeled by natural (or entire) numbers.

- Actions are transformations of the world into itself. So we may say that an action changes a state, meaning that the next element of the sequence is an image of the previous one under the action.

- In fact an action is a rule how to change the state when the action is terminated given a state when it is initiated. Therefore we may speak about tasks. A task is a set of functions from world into itself parameterized by a subset of the world. This subset is a preguard of a task (and of the function). A postguard is a subset of the world outside of which the function is an identity (the action fails to change the state).

- We are interested in the behaviour of the system, so we may want to decide which collections of tasks may give same (possible) sequences. This is done by introducing some equivalence relations between the tasks.

- On the other hand we are interested in some extremal set of tasks (minimal or maximal with respect to some criterion but still within the equivalence class). This may be done by reducing the tasks by rejecting the subtasks which are idle (i.e. acts as identities on the world). Similarly we may operate on the pre- and post-guard.

- In order to describe the behaviour completely we want (or have) to know what are the actions which may change the state at given moment (so called pending actions). To do this we introduce an abstract set called the set of agents and at each moment we distribute the actions among agents. Therefore the evolution of the system is a sequence of pairs consisting of a state and a distribution of actions.

- We may impose some restrictions on the distribution of the actions. For example we may want that an action persists until it is terminated i.e. until there was an attempt to implement it in the system. That means that for a given agent an action assigned to it at some moment (initiated) does not change in following moments until the system is changed by this action.

- If we are interested in all possible behaviors of the system then we may consider all possible sequences which are allowed by the rules of distribution and the rules of changing the states.

- There is a natural dynamic in the set of sequences, which is given by a shift. This corresponds to the evolution in time. If we say that some evolution is (time-)invariant we mean that some subset of such a collection is invariant under the shift.

- In the set of sequences we may introduce distance and probability based on distance and probability in the world. Thus we may speak of convergence, limits and attractors in the topological and measure–theoretical senses.

- For a finite world the theory of such sequences is called the theory of subshifts of finite type. In this theory one may describe such physical notions as energy, pressure or entropy.

In the following sections we shall present a model in full details. First we introduce the notions and explain them on some simple examples. Then we list (without proof) some properties. Some of them are straightforward consequences of definitions, other are easy to prove, but some of them are still rather conjectures and the proofs do not seem evident. Finally we state some notions from the subshift of finite type theory, and state possible directions worth (in our opinion) further investigations.

2 Tasks & Agencies

In this section we give an exact description of the *behavioral specification* in terms of an evolution in some phase space. Given a state (element of the world or the phase space) its possible evolutions are represented by sequences of the states (here indices are natural numbers and play the role of the time). Not all sequences are allowed, the limitations (or rules of the evolution) are given by a set of transformations called *tasks*. Saying that at time j' a transformation changes a state a into b we mean that for a given sequence (w_j) we have $w_{j'} = a$ and $w_{j'+1} = b$.

For each element of the sequence (or state at given moment of time) there are several *pending actions* or available transformations which may change the state i.e. which describe the rule binding the actual and next elements of the sequence. At the next element of the sequence there is also some new collection of pending action, which is a function valued in the set of tasks.

A step in the evolution (a shift in the sequence by one) consists in changing the state by one pending action (evolution of states) and choosing a new collection of pending actions, which differs form the old one in a specific way (evolution of actions).

Thus to describe the (possible) evolution we have to consider both states and pending actions and the rules of steps.

There are several conditions (*pre-* and *post-guards*) which limit the possible actions of tasks and the tasks themselves.

We are interested in the evolution of the world, i.e. given a state we observe its changes under some action. We do not assume that this evolution is determined by the initial state, therefore we have to consider many possible developments.

We allow one of several actions to take place. A state may be transformed by some action in progress, and new actions may be initiated.

The state may be changed by one of the previously initiated action, and the change happens if the state belongs to the *postguard* condition of the action. If the postguard conditions is not fulfilled there is no transformation (the transformation is an identity). On the other hand for each state there is a number of actions which may start (for any of these actions our state belongs to (or fulfills) the *preguard* condition of the action). By this we mean that if the distribution of pending actions for the next step differs from the previous one some old pending action may be lost, but new ones must have preguards fulfilled by the new state.

2.1 The notions

The model

We denote by $\mathcal{F}(\mathcal{A},\mathcal{B})$ the set $\mathcal{B}^{\mathcal{A}}$ of functions from $\mathcal{A}$ to $\mathcal{B}$.

- Let $\mathcal{W} \neq \emptyset$ be *the world.*

Tasks

- We say that $\mathcal{Z}_{P,Q} \subset \mathcal{F}(P \times \mathcal{W}, \mathcal{W})$ is *a task* with a preguard $P \subset \mathcal{W}$ and a postguard $Q \subset \mathcal{W}$ if for any function $Z \in \mathcal{Z}_{P,Q}$ we have $Z(p,w) = w$ for any $w \in \mathcal{W} \setminus Q$ and $p \in P$. Elements Z from the task $\mathcal{Z}$ will be called *subtasks.*
- For any $Z \in \mathcal{Z}_{P,Q}$ and $p \in P$ we have a map $Z_p : \mathcal{W} \to \mathcal{W}$ given by $w \mapsto Z(p,w)$ which we shall call *an action anchored at p.*
- We say that the action Z_p is *idle* if $Z_p = \mathrm{id}$.
  A subtask is idle if all actions Z_p are idle for $p \in P$.
  A task is *idle* if all its subtasks are idle, i.e. $Z_p = \mathrm{id}$ on $\mathcal{W}$ for all Z from the task and all p from the preguard.
- We say that a task is *busy* if there for any $Z \in \mathcal{Z}$ exists a $p \in P$ such that $Z_p \neq \mathrm{id}$ (it has no idle subtasks).
- We say that a task is *tight* if for any p in its preguard there exists a Z from the task such that $Z_p \neq \mathrm{id}$. (All elements from the preguard are important).
- We say that a task is *exhausting* if for any $q \in Q$ there exists a $Z \in \mathcal{Z}$ and $p \in P$ such that $Z_p(q) \neq q$.
- We say that a task is *demanding* if for any $Z \in \mathcal{Z}$ and $q \in Q$ there exists a $p \in P$ such that $Z_p(q) \neq q$.

Reducts

- A busy *reduct* of the task $\mathcal{Z}$ is a task $\mathcal{Z}'$ such that $P' = P$, $Q' = Q$ and $Z \in \mathcal{Z}'$ iff $Z \in \mathcal{Z}$ and Z is non-idle (i.e. $Z_p \neq \mathrm{id}$ for some $p \in P$).
- A tight reduct of the task $\mathcal{Z}$ is a task $\mathcal{Z}'$ such that $Q' = Q$, $p \in P'$ iff $p \in P$ and $Z_p \neq \mathrm{id}$ for some Z. Moreover $Z' \in \mathcal{Z}'$ iff there is a $Z \in \mathcal{Z}$ such that $Z'_p = Z_p$ for any $p \in P'$ and $Z_p \neq \mathrm{id}$ for some p.
- An exhausting reduct of the task $\mathcal{Z}$ is a task $\mathcal{Z}'$ such that $P' = P$ and $q \in Q'$ iff $q \in Q$ and $Z_p(q) \neq q$ for some $p \in P$ and $Z \in \mathcal{Z}$. Moreover $Z' \in \mathcal{Z}'$ iff there exists a non-idle $Z \in \mathcal{Z}$ (i.e. $Z_p \neq \mathrm{id}$ for some $p \in P$) such that $Z' = Z$ for each $p \in P$.
- A demanding reduct of the task $\mathcal{Z}$ is a task $\mathcal{Z}'$ such that $P' = P$ and $q \in Q'$ iff $q \in Q$ and for each non-idle $Z \in \mathcal{Z}$ there exits a $p \in P$ such that $Z_p(q) \neq q$. Moreover $Z' \in \mathcal{Z}'$ iff $Z' = Z$ for some non-idle $Z \in \mathcal{Z}$.

Projects

- *A project* $\mathcal{P}$ is a family of tasks
$$\mathcal{P} = \bigcup_{i \in I} \mathcal{Z}_i = \bigcup_{i \in I} \mathcal{Z}_{P_i, Q_i} \, .$$
- A project defines a set of *available* actions $\mathcal{Q} \subset \mathcal{F}(\mathcal{W}, \mathcal{W})$ given by
$$\phi \in \mathcal{Q} \text{ iff there exist } i \in I,\ Z \in \mathcal{Z}_i \subset \mathcal{P}, \text{ and } p \in P_i \text{ such that } \phi = Z_p \, .$$
- A project is *complete* if $\cup_{i \in I} P_i = \mathcal{W}$. A non-complete project may be completed by adding an (idle) task $\mathcal{Z}_0$ with one subtask Z_0 and with $P_0 = \mathcal{W} \setminus \cup_{i \in I} P_i$ and $\mathcal{Q}_0 = \emptyset$.
- A complete(d) project is busy (tight) if there is at most one idle task, which is of cardinality one, such that its preguard is disjoint with the preguards of other tasks which all are busy (tight).
- A reduct of a (complete-d) project is a (complete-d) project such that all its tasks but (maybe) one, which is idle and of cardinality one, are reducts (of an appropriate type).

Agencies and Managements

- Given a set $\mathcal{A}$ (called *the agency* or the set of *agents*) we define *a management* of a (complete-d) project $\mathcal{P}$ to be a set of functions $\mathcal{M} = \mathcal{F}(\mathcal{A}, \mathcal{Q})$.
  Each element of this set may be thought of as a decision how to distribute available actions from $\mathcal{Q}$ among the agents $A \in \mathcal{A}$.

- For an infinite agency (with an infinite number of agents) we assume always a infinite–to–one management, i.e. if $\alpha \in \mathcal{M}$, and $\alpha(A_0) = Z_p$ for some $A_0 \in \mathcal{A}$ then there are infinitely many agents A $\alpha(A) = Z_p$ (i.e. for any Z_p the cardinality of $\alpha^{-1}(Z_p)$ is either zero or infinity).

- We are only interested in the cardinality of an agency, i.e. in the number of agents. It is clear what is meant by *larger* or *smaller* agency.

Enterprises

- *An enterprise* $\mathcal{E}$ of an agency $\mathcal{A}$ realizing (or managing) a (complete-d) project $\mathcal{P}$ is a (maximal) subset of $\mathcal{E} \subset (\mathcal{W} \times \mathcal{M})^{\mathcal{N}}$ of sequences which fulfills the condition stated below. First we explain the structure of an enterprise.
  If $e \in \mathcal{E}$, then $e = (w_j, \alpha_j)_{j=0}^{\infty}$. For any $j \in \mathcal{N}$ we have $\alpha_j \in \mathcal{M}$, therefore for any $A \in \mathcal{A}$ the function $\alpha_j(A) \in \mathcal{Q}$ is an available map of the form Z_p for some $i \in I$, $Z \in \mathcal{Z}_i \subset \mathcal{P}$ and $p \in P_i$. Hence an agent may act on the world, $\mathcal{W} \to \mathcal{W}$, by $w \mapsto Z_p(w) = Z(p, w)$, which we may write $w \mapsto \alpha_j(A)(w)$.

  The condition reads : For any j there exists an $A_0 \in \mathcal{A}$ such that

  - $w_{j+1} = \alpha_j(A_0)(w_j)$ and
  - $\alpha_{j+1}(A_0) = Z_{w_{j+1}}$ for some $Z \in \mathcal{P}$. (It means implicitly that w_{j+1} fulfills the preguard of $Z \in \mathcal{Z} \subset \mathcal{P}$, this is always possible for some $\mathcal{Z}$, as the project was complete-d).
  - Moreover if $\alpha_{j+1}(A) \neq \alpha_j(A)$ (for some $A \neq A_0$ then $\alpha_{j+1}(A) = Z_{w_{j+1}}$ for some Z, i.e. new available actions assigned to the agents at this moment must be anchored at w_{j+1}.

- We say that an enterprise is *quiet* if the distribution of non-idle activities of agents α_{j+1} does not differ too much from the distribution α_j. By this we mean that if $w_{j+1} = \alpha_j(A_0)(w_j)$ and $A \neq A_0$ then if $\alpha_j(A) \neq \mathrm{id}$ then $\alpha_{j+1}(A) = \alpha_j(A)$.

  A short way to say it is that at the moment j the change in the distribution of available actions may occur only at idle actions and the action which just took place. The management takes as little decision as possible, summoning only agents which are not active and the agent which just finished its job.

- We say that the enterprise is *active* if for any agent with an idle activity $\alpha_j(A) = \mathrm{id}$ we have $\alpha_{j+1}(A)$ is not idle if possible. By this we mean that if w_{j+1} belongs to a preguard P_i, whose task is not idle then $\alpha_{j+1}(A) \neq \mathrm{id}$ for $A \in \mathcal{A}$.

Developments

- The projection of an enterprise on $\mathcal{W}^{\mathcal{N}}$ is called *a development.* $\mathcal{D} = \{(w_j)_{j=0}^{\infty} \,:\, \exists\, e \in \mathcal{E}\ e = (w_j, \alpha_j)_{j=0}^{\infty}\}$ for some sequence of α_j.

- An element of a development (i.e. a sequence (w_j)) is called *a path.*

- Given the world $\mathcal{W}$ we say that the enterprise $\mathcal{E}$ is *wider* than $\mathcal{E}'$ if $\mathcal{D}' \subset D$.

- We say that

  - a task $\mathcal{Z}_{P,Q}$ is *broader* then $\mathcal{Z}'_{P',Q'}$ if for any $Z' \in \mathcal{Z}'$ and $p \in \mathcal{P}'$ such that $Z'_p \neq \mathrm{id}$ (Z' is not idle)we have $p \in P$ and there is a $Z \in \mathcal{Z}$ such that $Z_p = Z'_p$ on $\mathcal{W}$,
  - $\mathcal{Z}$ is *an extension* of $\mathcal{Z}'$ if $P' \subset P$ and for any $Z' \in \mathcal{Z}'$ such that $Z'_{p'} \neq \mathrm{id}$ for some $p' \in P'$ (Z' is not idle) there exists a $Z \in \mathcal{Z}$ such that for any $p \in P'$ we have $Z'_p = Z_p$,
  - a project $\mathcal{P}$ is *broader* then $\mathcal{P}'$

    if for any non-idle $Z' \in \mathcal{Z}'_{i_1} \subset \mathcal{P}'$ there is a $Z \in \mathcal{Z}_{i_2} \subset \mathcal{P}$ such that for any $p \in P'_{i_1}$ we have either $Z_p = \mathrm{id}$ or $p \in P_{i_2}$ and there is a $Z \in \mathcal{Z}_{i_2}$ such that $Z_p = Z'_p$ on $\mathcal{W}$,

- $\mathcal{P}$ is *an extension* of $\mathcal{P}'$ if for any $Z' \in \mathcal{Z}'_{i_1} \subset \mathcal{P}'$, Z' non-idle there is a $Z \in \mathcal{Z}_{i_2} \subset \mathcal{P}$ such that $P'_{i_1} \subset P_{i_2}$ and for any $p \in P'_{i_1}$ we have $Z_p = Z'_p$.

The notion being broader means that for any available non-idle action generated by one task there is the same available action generated by the other one. In the extension all non-idle elements from one task have their counterparts in the second one. Shortly in the second case we require that there is a correspondence between non-idle elements of the tasks, while in the first one the correspondence occurs only at the level of available actions.

- We say that an enterprise $\mathcal{E}$ is developing *faster* than $\mathcal{E}'$ if for any sequence $(w') \in \mathcal{D}'$ and $j' > 0$ there exists a sequence $(w) \in \mathcal{D}$ and $j < j'$ such that for any $l \geq 0$ we have $w'_{j'+l} = w_{j+l}$. That means that any subsequence from $\mathcal{D}'$ appears sooner in $\mathcal{D}$.

- We use the word *eventually* in front of bigger or quicker if there exists an N such that these relations are restricted for the all sequences after rejecting first N elements.

- The subset $W \subset \mathcal{W}$ is N-invariant in the development $\mathcal{D}$ if for any $(w) \in \mathcal{D}$ such that if for some j' $w_j, w_{j+1}, \ldots, w_{j+N} \in W$ then $w_j \in W$ for all $j > j'$.

- *A shift* τ is a transformation on the set of sequences defined by $\tau(w_j) = w_{j+1}$, i.e. the image of a sequence under shift is the sequence with truncated first element.

- The subset $D \subset \mathcal{D}$ is *shift-invariant* if $(w_j)_{j=0}^{\infty} \in D$ then $(w_{j+1})_{j=1}^{\infty} \in D$.

- *A strict attractor* is a shift-invariant set $\mathcal{M}$ such that its *bassin* $\mathcal{B} = \bigcup_{n=0}^{\infty} \tau^{-n}\mathcal{M} \neq \mathcal{M}$, and it is minimal (any invariant subset of $\mathcal{M}$ has smaller bassin).

Irreducibility

- We say that the task $\mathcal{Z}$ is *decomposed* into tasks $\mathcal{Z}'$ and $\mathcal{Z}''$ if P is a disjoint union $P' \cup P''$ and for each $Z \in \mathcal{Z}$ there are $Z' \in \mathcal{Z}'$ and $Z'' \in \mathcal{Z}''$ such that $Z_p = Z'_p$ if $p \in P'$ and $Z_p = Z''_p$ if $p \in P''$. The decomposition is not trivial if both sets P' and P'' are non-empty and both tasks $\mathcal{Z}'$ and $\mathcal{Z}''$ are not idle.

- We say that a project may be *decomposed* if there is a decomposition of $\mathcal{W}$ into two (or more) non-empty disjoint subsets $\mathcal{W}' \cup \mathcal{W}''$ and each task of the project may be decomposed by $P \cap \mathcal{W}'$ and $P \cap \mathcal{W}''$ and at least one of these decomposition is not trivial.
- A project is *irreducible* if there is no decomposition of the project with the property $Z(P \cap \mathcal{W}', \mathcal{W}') \subset \mathcal{W}'$ and $Z(P \cap \mathcal{W}'', \mathcal{W}'') \subset \mathcal{W}''$ for all $Z \in \mathcal{P}$.

Simplification

- We say that a sequence is *simple* if the equality $w_j = w_{j+1}$ for some j implies $w_i = w_j$ for all $i > j$.
- *A simplification* of a sequence is its maximal simple subsequence. We may think of it as of a map $s : \mathcal{W}^{\mathcal{N}} \to \mathcal{W}^{\mathcal{N}}$ such $s(w_j) = w_{k_j}$ with $k_0 = 0$ and if k_j is defined then either $k_{j+1} = i$ is minimal $i > k_j$ such that $w_i \neq w_{k_j}$ or if $w_i = w_j$ for all $i > j$ then $k_{j+1} = k_j + 1$.
- We say that two sequences are *simply the same* if they have same simplifications

Example 2.1 *Consider the world* $\mathcal{W} = \{0,1\}^N$ *for some* $N \in \mathcal{N}$. *Let* $\mathcal{P} = \cup_{i=0}^{N+1}\{Z_i\}$ *where for* $i = 1, \dots, N$ *we have* $P_i = \{w \in \mathcal{W} : w_1 = 0\}$, $Q_i = \{w \in \mathcal{W} : w_j = 1, j < i\}$ *and* $Z_i(p,w)_j = 1$ *for* $j \leq i$ *and* $Z_i(p,w)_j = 0$ *for* $j > i$. *For* $i = 0$ *we set* $P_0 = \{(1,\dots,1)\}$, $Q_0 = \mathcal{W}$ *and* $Z_0(p,w) = (0,\dots,0)$. *For* $i = N+1$ *we set the preguard* $P_{N+1} = \mathcal{W} \setminus \cup_{i=0}^{N} Z_i(\cdot, \mathcal{W})$, *postguard* $Q = \mathcal{W}$ *and* $Z(p,w) = (0,\dots,0)$. *We complete the project in a standard way.*

This project is ending in idle actions if the agency has cardinality smaller than N.

Example 2.2 *Let* $\mathcal{W} = \{0,1,2,3\}$. *For* $i = 1,2$ *let* $\mathcal{Z}_i = \{Z_i\}$ *with* $Z_i(p,w) = 0$ *for* $w = 0, i$, $Z_i(p,3) = i$ *and* $Z_i(p, 3-i) = i$. *If* $Q_i = \{0\}$ *then the task* $\mathcal{Z}_i$ *is idle. If* $0 \notin Q_i \neq \emptyset$ *then the task is tight. If the project consists only of one of these tasks then it lands at* 0 *after at most three steps. If the project includes both of them then eventually periodic sequences as* 312121212... *appear in the development.*

2.2 Properties

- The situation when a task is neither idle nor busy is not excluded.
- $\mathcal{Z}_{\emptyset,Q} = \emptyset$ by definition.
- $\mathcal{Z}_{P,\emptyset}$ is idle.
- Being broader is reflexive and transitive.
- Being extension is reflexive and transitive.
- Being an extension implies being broader.
- The relation $\mathcal{Z}$ is broader than $\mathcal{Z}'$ and $\mathcal{Z}'$ is broader than $\mathcal{Z}$ is an equivalence relation between tasks in the same world.
- For any non-idle task there is an equivalent (in the sense of broader) busy, tight task.
- The relation of being mutual extensions is an equivalence.
- For any non-idle ask there is an equivalent (in the sense of extension) busy task.
- Similarly with projects.
- Being tight and being busy are extremal conditions on the preguard.
- Being exhausting and being demanding are extremal conditions on the postguard.
- Tasks are equivalent to their reducts.
- Consecutively taking reducts is commutative for different types and idempotent for the same type.
- Reducts are extremal with respect to their type (e.g. a tight reduct is minimal (with no subtask and no task with smaller preguard) tight task equivalent (and therefore satisfying a maximality condition) to the task.

- Equivalent projects realized with the agencies of the same number of agents have same developments.
- Bigger agency realizes bigger development of the same project. (Both possibility strictly and not-smaller are not *a priori* excluded).
- All infinite agencies realize same development of a given project.
- For finite agencies the development depends only on the cardinality of the agency (i.e. on the number of agents).
- For any path $w_j \in \mathcal{D}_\infty$ of a development realized by a infinite agency there is a finite agency realizing the same path.
- If the project is not irreducible then there is a decomposition of its developments.

3 Subshifts of finite type

If we suppose that the world $\mathcal{W}$ is finite and the project $\mathcal{P}$ is finite then we may use the modeling *via* the theory of subshifts of finite type.

Let $\mathcal{K}$ be a finite set and $\mathcal{X} = \mathcal{K}^{\mathbf{N}}$ be the set of (one-sided) sequences valued in $\mathcal{K}$. The dynamical system $(\mathcal{X}, \sigma)$ is called a full shift, where the shift σ acts on the sequence (a_n) by skipping its first element, $\sigma((a_n)_{n=0}^{\infty}) = (a'_n)_{n=0}^{\infty}$ with $a'_n = a_{n+1}$.

Now let $T = (t_{m,n})$ be a $k \times k$ matrix, $k = \mathrm{card}\mathcal{K}$, and $t_{m,n} \in \{0, 1\}$. We define a subspace $\mathcal{X}_T \subset \mathcal{X}$ by the condition $(a_n) \in \mathcal{X}_T$ iff for any n $t_{a_n, a_{n+1}} = 1$. We can describe the sequences from $\mathcal{X}_T$ as the sequences for which the possible successors $a' = a_{n+1} \in \mathcal{K}$ (in the sequence) of the element $a = a_n \in \mathcal{K}$ are described by the permission matrix T. The matrix is called irreducible if there is some m such that T^m has all entries strictly positive. It means that after m steps we may go from any state to any state. The entries of T^m, which are obviously natural numbers, represent the number of paths joining two states in m steps.

Example 3.1 *Let $\mathcal{K} = \{1, 2\}$ and*

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

then $\mathcal{X}$ is the set of all sequences of 1 and 2 and $\mathcal{X}_T$ are the sequences with no two consecutive 2's.

If $\mathcal{K} = \{1, 2, 3\}$ and

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

then $\mathcal{X}_T$ is the set of all sequences of 1,2 and 3 where 3 must be followed by 2 and preceded (if not staying at the beginning of the sequence) by 1.

The dynamical system $(\mathcal{X}_T, \sigma)$ is called the subshift of finite type. For more information see [Bow75]. In this system the following notions are naturally defined.

- **The metric.** Let $\tilde{d}$ be the discrete metric on the set $\mathcal{K}$. Then we can define the metric d on $\mathcal{X}$ (and on $\mathcal{X}_T$ as well) by $d((a_n),(b_n)) = \sum \tilde{d}(a_n, b_n)/2^n$. The balls consist of sequences with same values of in given finite number of initial coordinates. Two sequences are close if their sufficiently long initial parts are identical.

- **The topology** is the metric topology in the above sense. The base of neighbourhoods are the so called cylinders, i.e. the sets with fixed values on some finite coordinates. The cylinders are the unions of finite number of balls.

- **The measure.** Let $(p_i)_{i=1}^k$ be a probability distribution on the set $\mathcal{K}$. We may define the probability of the cylinder by the product of p_i's corresponding to the fixed values defining the cylinder. This definition expands (*via* Kolmogorov's theorem) to the measure on Borel sets in $\mathcal{X}$. The measure on $\mathcal{X}_T$ may be (with some care) derived by taking the conditional measure.

We want to describe the evolution of a system with acting agents as a sequence of states. One sequence represents one possible development of a system. The state is a configuration of the system including *the real world* and *the agents with their private worlds, flags of actual activities and preguard and postguard information.*

The idea borrowed from the *subshifts of finite time* is to reduce the space of all sequences of configurations (states) to the sequences for which only some successors of a state are possible. This corresponds on one hand to the conditions making *actions acceptable* and on the other to the permission matrix. We shall be more specific in the next section.

Let us now consider in detail an easy example. The goal of this example is to present a system small enough to make the full description readable but which has the features leading to more complicated applications as a system with a flip-flop and a counter controlled by several agents.

Example 3.2 (An agent copies the flip-flop into a register) *Consider the system S with a flip-flop x and a register r. The flip-flop changes the value from 0 to 1 and from 1 to 0 independently from the rest of the system. The aim of the acting agent A is to put in r the value of x in case when the actual state of the flip-flop is different from the one remembered by the agent. The agent has two registers : a working one w and an activity one a.* □

Let $\mathcal{K}$ be the set of following vectors $(S, A) = ((x,r),(w,a))$ with $x, r, w, a \in \{0,1\}$. The cardinality of $\mathcal{K}$ is $2^4 = 16$. A 0-1 matrix $T = (t_{V,V'})$ describes the admissible followers $V' \in \mathcal{K}$ of a state V in the sequence of the evolution of the system (S, A).

Instead of writing the 16×16 matrix T we shall describe the permissions, the entry of the matrix is 1 iff the following conditions are fulfilled :

1. $((x,r),A)$ can follow and be followed by $((x',r),A)$ for any choice of x, x' and r, A, i.e. $t_{((x,r),A),((x',r),A)} = 1$. This is the independence of the flip-flop.

2. The value of w is changed into the value of x when some condition $G(S)$ (*a preguard* from [Tur94]) is fulfilled, then also a changes from 0 to 1 (the agent becomes active). In other words under $G(S)$ for $V = ((x,r),(w,0))$ and $V' = ((x',r),(x,1))$ we have $t_{V,V'} = 1$, for any x, x', w and r. In our case G is fulfilled by any state S (there is no *preguard* condition).

3. When $G(S)$ is fulfilled and $a = 0$ then the only possible follower of V is V' with $a' = 1$ and $w' = x$.

4. a may change from 1 to 0 only if w changes to $w' = f(w)$ and y changes to $F(S, A)$. f is a predefined function depended of the aim of the agent, in our case $f(w) = w$, and $F(S, A) = r$ or $f(w)$ depending

if some condition (*a postguard* from [Tur94]) is fulfilled. In our case $F(S, A) = f(w)$ if $f(w) = w \neq x$ and y when $f(w) = w = x$. That means that $t_{V,V'} = 1$ for

(a) $V = ((0, r), (1, 1))$ and $V' = ((x', 1), (1, 0))$ or

(b) $V = ((0, r), (0, 1))$ and $V' = ((x', r), (1, 0))$ or

(c) $V = ((1, r), (1, 1))$ and $V' = ((x', r), (1, 0))$ or

(d) $V = ((1, r), (0, 1))$ and $V' = ((x', 0), (0, 0))$

5. We do not permit the register r to change in a different way than by converting it into the value of w as described above at (4).

6. We do no say what happens to w when a stays equal to 0 (this situation is in our case excluded by (2,3)), but when $a, a' = 1$ w cannot change. When $a = 1$ and $a' = 0$ there is no condition on w.

Here is the whole permission matrix T. We represent the state by the hex-digit equal to $x + 2r + 4w + 8a$. The double lines separates different activities ($a = 0$ and $a = 1$) and single lines different w. We mark only the entries equal to 1. Repeated double ones say that the flip-flop may change its state in an independent way.

The first quadrant ($a = 0, a' = 0$ of the matrix is empty because in our example the system must change from idle to active (condition 3) in particular because there is no restriction given by a *preguard*, as $G(S)$ is always fulfilled. The second quadrant ($a = 0, a' = 1$) shows the copying of x into w' while leaving r. The third one ($a = 1, a' = 0$) shows what happens when the agent terminated its job. There are two pairs of entries in each row due to the fact that the new value of w' is not determined. The register r changes only in rows B and D. The fourth quadrant shows that (up to the flip-flop changes) the system stays identical when the agent is busy.

Let us take now the set $\mathcal{X}$ of all (one-sided) sequences of symbols $0 \ldots F$, and $\mathcal{X}_T$ the subshift derived from $\mathcal{X}$ with the matrix T. The sequences from $\mathcal{X}_T$ represent all possible evolutions of the system which fulfill the rules of permission. We can now compare two evolutions, try to find an invariant measure then attractors (in both topological and metrical sense).

$V \setminus V'$	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0									1	1						
1													1	1		
2											1	1				
3															1	1
4									1	1						
5													1	1		
6											1	1				
7															1	1
8	1	1			1	1			1	1						
9	1	1			1	1			1	1						
A			1	1			1	1			1	1				
B	1	1			1	1					1	1				
C			1	1			1	1					1	1		
D	1	1			1	1							1	1		
E			1	1			1	1							1	1
F			1	1			1	1							1	1

Table 1: The permission matrix for a system copying a flip-flop. Explanation in the text.

4 The thermodynamics

There is already a very well developed theory of subshift of finite type. In this section we point out how the notions of thermodynamics may be used in the investigations of dynamical systems consisting of sets of sequences as the phase space and a shift as the transformation. This is based on the books [Bow75, Rue78].

4.1 Probability measures, introductory notions.

Let $(X, \mathcal{B}, \mu)$ be a *probability space*, i.e. $\mathcal{B}$ is a σ-field of subsets of X (called *measurable* sets) and μ is a nonnegative *measure* on $\mathcal{B}$ with $\mu(X) = 1$. Usually we work with a fixed transformation of the space. We want this transformation to preserve the measure (or the measure to be invariant with respect to the transformation).

An *automorphism* is a measurable bijection $T : X \to X$ (i.e. $T^{-1}E \in \mathcal{B}$ iff $E \in \mathcal{B}$) for which $\mu(T^{-1}E) = \mu(E)$, $E \in \mathcal{B}$. An *endomorphism* of a probability space is a measurable transformation such that $\mu(T^{-1}E) = \mu(E)$, $E \in \mathcal{B}$.

In case when X is a compact space and T is a homeomorphism (or a continuous map) one usually sets $\mathcal{B}$ to be the family of Borel sets. The measure is called then a Borel probability measure. Let $M(X)$ be the set of Borel probability measures and $M_T(X)$ its subset of measures which are invariant with respect to T. We have by definition $\mu \in M_T(X)$ iff $\mu \circ T^{-1} = \mu$. For any $\mu \in M(X)$ we can define $T^*(\mu) = \mu \circ T^{-1}$.

Real-valued continuous functions $C(X)$ on the compact metric space X form a Banach space with the norm $\|f\| = \sup_{x \in X} |f(x)|$. The weak $*$-topology on the space $C(X)^*$ (i.e. the space of continuous linear functionals $\alpha : C(X) \to \mathcal{R}$) is generated by the sets of the form $U(f, \epsilon, \alpha_0) = \{\alpha \in C(X)^* : |\alpha(f) - \alpha_0(f)| < \epsilon\}$ with $f \in C(X)$, $\epsilon > 0$, $\alpha_0 \in C(X)^*$.

Riesz Representation. For each $\mu \in M(X)$ define α_μ by $\alpha_m u(f) = \int f \, d\mu$. Then $\mu \to \alpha_\mu$ is a bijection between $M(X)$ and $\{\alpha \in C(X)^* : \alpha(1) = 1 \text{ and } \alpha(f) \geq 0 \text{for} f \geq 0\}$. We identify α_μ and μ. We call the weak topology on $M(X)$ the topology induced by this identification from the weak $*$–topology on $C(X)^*$.

We have following properties of the spaces $M(X)$ and $M_T(X)$.

- *$M(X)$ is a compact, convex, metrizable space.*
  This follows from the fact that the weak topology on $M(X)$ is equivalent with the topology induced by the metric $d(\mu,\nu) = \sum_{n=1}^{\infty} | \int f_n \, d\mu - \int f_n \, d\nu| \cdot \|f_n\|^{-1}/2^n$, where (f_n) is a dense subset of $C(X)$.

- *$M_T(X)$ is a nonempty closed set of $M(X)$.*
  T^* is a homeomorphism of $M(X)$ and $M_T(X) = \{\mu \in M(X) : T^*(\mu) = \mu\}$. For $\mu \in M(X)$ let ν be an accumulation point from $\frac{1}{n}\sum_{k=0}^{n-1}(T^*)^k\mu$. Then ν is T invariant.

- *$\mu \in M_T(X)$ iff $\int (f \circ T) \, d\mu = \int f \, d\mu$ for all $f \in C(X)$.*
  This is Riesz representation theorem applied to $T^*\mu = \mu$.

Suppose that $A = (a_{ij})$ is a $n \times n$ matrix of nonnegative integers. We may consider a (closed) subset Σ_A (resp. Σ_A^+) of Σ (resp. Σ^+) consisting of the sequences $\mathbf{x}$ such that $a_{x_k x_{k+1}} > 0$ for any k. We may assume that A is such that for any $k \in F$ there is an $\mathbf{x} \in \Sigma_A^{(+)}$ with $x_0 = k$, otherwise one can take an $m \times m$ matrix B with $m < n$ and $\Sigma_A^{(+)} = \Sigma_B^{(+)}$. These sets with a shift transformations are subshifts of finite type.

Let us state the following result. The shift τ is topologically mixing (i.e. for any U, V nonempty open subsets of $\Sigma_A^{(+)}$ there is an N such that $\tau^M U \cap V \neq \emptyset$ for $M > N$) iff $A^M > 0$ (i.e. all entries are strictly positive) for some M.

In the set $C(\Sigma_A^{(+)})$ there is a special of continuous real-valued functions on $\Sigma_A^{(+)}$ where is a special family $\mathcal{F}_A$ of functions with positive Hölder exponent with respect to the metric d_γ. $\phi \in \mathcal{F}_A$ iff $\mathrm{var}_k(\phi) := \sup\{|\phi(\mathbf{x}) - \phi(\mathbf{y})| : x_i = y_i \mathrm{for} |i| \leq k\} \leq b\alpha^k$, for some $b > 0$ and $\alpha \in (0,1)$.

4.2 Gibbs measures

Suppose that a 'physical system' has possible states $1, \ldots, n$ and the energies of these states are $E_1, \ldots, E_n$. Suppose further that this system is not isolated but in permanent contact with a 'large heat source' which remains at the constant temperature T. Therefore the total energy of the system is not fixed and any state of the system may actually occur. There is a following 'physical' fact. The probability p_j that the system is at state j is given by *Gibbs distribution* $p_j = \exp(-\beta E_j)/\sum_i \exp(-\beta E_i)$, with $\beta = 1/kT$, k a 'physical' constant.

This is connected to the following 'mathematical' fact. Given real numbers $a_1, \ldots, a_n$ the function $F(p_1, \ldots, p_n) = \sum_i (a_i - \log p_i) p_i$ attains in the simplex $\sum_i p_i = 1$, $p_j \geq 0$ a maximum $\log \sum_i \exp(a_i)$ at the point $p_j = \exp(a_j) / \sum_i \exp(a_i)$. The quantity $h(p_1, \ldots, p_n) = \sum_i -p_i \log p_i$ is called the *entropy* of the distribution (p_i). We assume $0 \log 0 = 0$ for $x \log x$ to be continuous at $x = 0$.

If we put $a_i = -\beta E_i$ then we have $\sum a_i p_i = -\beta E$ with average energy E and the Gibbs distribution *maximizes* $S - \beta E$, where S stays for entropy. The *minimized* quantity $P = E - kTS$ is called *free energy*. Therefore the principle reads 'nature maximizes the entropy' when the energy is fixed but 'nature minimizes the free energy' when the energy is not fixed. One can generalize such a distribution to the system Σ_A.

Let $\phi : \Sigma_A \to \mathcal{R}$ be Hölder continuous. Then there is a unique invariant measure $\mu \in M_\tau(\Sigma_A)$ for which one can find constants c_1, c_2 and P such that $c_1 \leq \mu\{\mathbf{y} : y_j = x_j, j = 0, \ldots, m-1\} / exp(-Pm + \sum_{j=0}^{m-1} \phi \circ \tau^j(\mathbf{x})) \leq c_2$. One can call $P = P(\phi)$ a pressure of ϕ. The exact definition of the pressure is more complicated.

For such generalized 'Gibbs distribution' μ one has the *Variational Principle.* $h(\mu) + \int \phi \, d\mu = P(\phi)$.

The measure satisfying the above principle is called an *equilibrium state.* In one-dimensional lattice the equilibrium states for Hölder continuous functions are (unique) Gibbs distributions.

We can call $\phi \in C(X)$ observables, and $\mu \in M(X)$ states in the sense that $\mu(\phi) = \int \phi \, d\mu$ is the average value of ϕ in the space.

The configurations (i.e. the sequences $\mathbf{x} \in \Sigma_A$) can describe possible evolutions of the system. The matrix A describes *pre-* and *post-guards.* If we are interested in asymptotical behaviour of the limit sets we may define an observable ϕ and see what is a state μ which realizes the variational principle. This measure μ shows what configurations are 'important' from the point of view of the observable ϕ.

Now there is a couple of notions which needs to be interpreted in the setting of the behavioral specifications. We may want to know what are *energy, pressure, temperature* ? We may want to look for a 'good' potential energy, so that we see 'interesting' sequences with large probabilities. We may want to understand what are equilibrium states μ and observables ϕ ? Are we interested at all in the infinite setting ? Should we rather concentrate in the finite (but large) systems ?

The easiest case is when one considers as $\phi(\mathbf{x}) = -d_\gamma(\mathbf{x}, S)$, the distance of the configuration to a given set $S \subset \Sigma_A$. In the case of an *algorithm* — by this I mean that we are interested in an evolution of finite number of steps, i.e. finite iterations of the shift — this set S should be (forward) shift invariant or even a fixed point of the shift. The function ϕ cannot be split into two parts, describing the energy of the site 0 (due to x_0) and the potential energy of interactions between the site 0 and sites j for $j \in \mathcal{N}$. Nevertheless it describes in a sense how 'far' the actual configuration is from the desired form. Natural candidate equilibrium measures (states) should have supports on S (the integral part is then 0) and spread equally on S to maximize the entropy. If the set S is not forward invariant then the (invariant) measure cannot be supported only on S.

On the other hand the 'potential energy' may also describe the preferences of the observer, e.g. a weighted distance to some disjoint sets of acceptable 'final' configurations.

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